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# The instability of a fluid layer with time-dependent heating

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The problem of the stability and growth of disturbances in a fluid with time-dependent heating is investigated. The analysis is restricted to the case when the temperature gradient is large in a layer which is narrow by comparison with the overall depth of the fluid. An approximate method of solution is presented. Results of computations are also presented which illustrate the method of solution and the essential features of the problem.

## 1. Introduction

The present investigation is a study of the stability and growth of disturbances in a fluid layer in which the temperature gradient may be a function of depth and time. This particular problem was suggested by a crude experiment which was performed in order to gain some understanding of the temperature changes and convective motions near the surface of the ocean.

The experiment was as follows. A small tank, approximately 10 cm deep, 10 cm wide, and 20 cm in length, was filled with water and covered. The tank was left standing until all convective currents had decayed and the water was at a uniform temperature throughout. The cover was then taken off. At this time the water began to cool by evaporation. The temperature was measured at various depths by thermistors. The surface temperature was measured by a radiometer. The density gradients were observed by a Schlieren system. By use of this Schlieren system, a dark narrow band near the surface was observed initially. This band increased in thickness with time. After 15 or 20 sec, when the dark layer was several mm thick, small drops began to appear on the lower side of this layer. These drops began to grow and, after approximately 60 sec, broke away from the upper layer, gradually fell into the warmer water below, and diffused as they fell to the bottom. The formation and breaking away of these drops continued at irregular intervals.

Radiometer readings showed that the surface was cooled approximately 0·5° C by the evaporation. The effective depth (the depth at which the temperature difference decreased to $1/e$ of its surface value) to which this cooling penetrated varied with time, of course. After 15 sec, the effective depth was approximately 3 mm and after 60 sec it was about 7 mm.

The measurements were crude since this was just an exploratory experiment. However the following points can be made: (*a*) the temperature before instability began, or at least before the disturbance became large, changed rapidly with depth in a narrow layer near the surface; (*b*) the disturbances grew rapidly in time by comparison with the rate at which the unperturbed temperature (the temperature in the absence of convection) changed.

Questions which presumably may be asked in connexion with the initial formation and growth of the drops are: under the above conditions, (*a*) when does instability begin, and (*b*) how do initial disturbances grow with time?

The present analysis does not attempt to duplicate the exact conditions of the above experiment, but is a general investigation of the stability and growth of disturbances in a fluid layer characterized by a temperature gradient which is large in a layer narrow by comparison with the depth of the fluid and in which disturbances grow rapidly by comparison with changes in the unperturbed temperature. This type of analysis is important not only in understanding phenomena near the surface of oceans but also in stellar atmospheres (Gribov & Gurevich 1957) and in re-entry heating problems (Goldstein 1959).

Although much work has been done on the problem of the stability of a fluid layer, the restrictions and approximations have been such that they are not directly applicable to the present problem. The earliest experimental investigations were by Thomson (1882) and Bénard (1900). The early theoretical work was done by Rayleigh (1916), Jeffreys (1926, 1928), Low (1929), and Pellew & Southwell (1940). These investigations were chiefly concerned with the stability of a fluid layer in which the unperturbed state was characterized by a uniform temperature gradient. Moreover, these investigations were restricted to the questions of the onset of instability and the conditions at marginal stability. This work has been reviewed by Ostrach (1957) and Chandrasekhar (1961).

More recent work has been concerned with the stability of a fluid layer when the temperature gradient is not constant. In addition, the question of the rate of growth of disturbances has been examined. Morton (1957) has investigated the growth of disturbances when the temperature gradient was a slowly varying function of depth but independent of time. Gribov & Gurevich (1957) have obtained approximate limiting solutions for the onset of instability for the problem of an unstable layer with a uniform temperature gradient bounded by a stable layer. Goldstein (1959) has analysed the stability of a fluid layer with time-dependent heating. However, his method of solution consists of an analysis by Fourier series and is restricted to the case when the temperature is a slowly varying function of depth. These last three papers will be discussed more extensively in the following sections in connexion with the present investigation.

A general analysis of the present problem and the approximations involved are presented in the following section. The results of computations and a discussion of these results are given in § 3.

## 2. General theory

Consider a fluid layer bounded above and below but infinite in the horizontal direction. The boundary conditions at the upper and lower surfaces are such that the temperature of the fluid in the unperturbed state is dependent on both depth and time but constant in the horizontal direction. The depth $z^*$ will be measured vertically downwards from the upper surface. The co-ordinates $x^*$ and $y^*$ lie in the horizontal plane. The thickness of the layer is $d$.

It is assumed that the solution for the unperturbed state, the state with no convective motion, is known. In order to determine the solution when convective motions are present, although small so that squares and products of the velocities and temperature may be neglected, the perturbation equations for the velocities and temperatures must be obtained. These perturbation equations have been derived for the case when the unperturbed temperature is an arbitrary function of depth and time by Goldstein and the approximations involved are discussed there. These equations are presented below.

In the derivation, it is assumed that the dependence of all perturbation quantities on $x^*$ and $y^*$ has the form $\exp(ik_x x^* + ik_y y^*)$ where $k^2 = k_x^2 + k_y^2$ is the wave-number of the disturbance. The relationship between the periodic solutions implied by the above relations and the cellular patterns observed experimentally is discussed by Stuart (1964). The perturbation temperature $T^*$ and the perturbation vertical velocity $w^*$ can be written as

$$T^* = T_0\theta(z^*, t^*)\exp(ik_x x^* + ik_y y^*), \tag{1}$$

$$w^* = (\kappa/d)\,W(z^*, t^*)\exp(ik_x x^* + ik_y y^*), \tag{2}$$

where $T_0$ is a reference temperature, $\kappa$ is the thermal diffusivity, $t^*$ is the time, and $\theta$ and $W$ are dimensionless quantities proportional respectively to the perturbation temperature and vertical velocity.

A single equation governing $W$ can be written as

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)\left(\frac{\partial^2}{\partial z^2} - a^2 - \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial z^2} - a^2 - P\frac{\partial}{\partial t}\right)W = -\beta W, \tag{3}$$

where $z = z^*/d$, $a = kd$, $t = \nu t^*/d^2$, $P = \nu/\kappa$ and is the Prandtl number, $\nu$ is the kinematic viscosity, and $\beta$ is a dimensionless temperature gradient given by

$$\beta = -(g\alpha a^2 d^4/\kappa\nu)\,\partial\overline{T}^*/\partial z^*, \tag{4}$$

where $g$ is the acceleration due to gravity, $\alpha$ is the coefficient of thermal expansion, and $\partial\overline{T}^*/\partial z^*$ is the temperature gradient in the undisturbed fluid. An alternate form of (3) can be written as

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)^3 W - (P+1)\left(\frac{\partial^2}{\partial z^2} - a\right)^2\frac{\partial W}{\partial t} + P\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial^2 W}{\partial t^2} = -\beta W. \tag{5}$$

Auxiliary equations which are useful in the analysis are

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)\left(\frac{\partial^2}{\partial z^2} - a^2 - \frac{\partial}{\partial t}\right)W = F \tag{6}$$

and

$$\left(\frac{\partial^2}{\partial z^2} - a^2 - P\frac{\partial}{\partial t}\right)F = -\beta W, \tag{7}$$

where

$$F = (g\alpha T_0 a^2 d^3/\kappa\nu)\,\theta. \tag{8}$$

### 2.1. *Reduction to problem with static temperature gradient*, $\beta = \beta(z)$

If the unperturbed temperature gradient $\beta$ were independent of time and a function only of $z$, the solution of (3) could immediately be written as

$$W(z,t) = \omega(z)\, e^{\sigma t}, \tag{9}$$

where $\sigma$ is a constant. An ordinary differential equation for $\omega(z)$ would then be obtained which would lead to an eigenvalue problem which can be solved in a straightforward manner.

The observation that disturbances grow rapidly by comparison with changes in the unperturbed temperature indicate that a first approximation to (3), when $\beta = \beta(z,t)$, could be obtained by assuming that the rate of growth of disturbances at each instant of time depended only on the temperature gradient at that instant. This conjecture leads to an asymptotic method of solution of (3) in which the first approximation is simply that described above. The method is analogous to asymptotic approximations devised for ordinary differential equations (Jeffreys 1962).

Equation (9) suggests the substitution

$$W(z,t) = \omega(z,t)\, e^{\alpha\phi(t)}, \tag{10}$$

where $\alpha$ is a large parameter. If this is substituted into (3), or more conveniently into (5), and only the highest powers of $\alpha$ retained in each term, one obtains

$$(\partial^2/\partial z^2 - a^2)^3\,\omega - \sigma(P+1)\,(\partial^2/\partial z^2 - a^2)\,\omega + \sigma^2 P(\partial^2/\partial z^2 - a^2)\,\omega = -\beta\omega, \tag{11}$$

where $\sigma = \alpha\, d\phi/dt$. Since no derivatives of $\omega$ with respect to time appear in the above expression, $t$ can be considered as a parameter. The above equation can then be considered as an ordinary differential equation for $\omega$ with derivatives only with respect to $z$.

To this approximation, the problem has now been reduced to solving (11) at successive times $t$ in order to find $\omega(z,t)$ and $\sigma(t)$. This will be done in the following section. Once $\sigma$ is known, $\phi$ can be found from

$$\alpha\phi = \int_0^t \sigma\, dt \tag{12}$$

and therefore $W$ can be found from (10).

By retaining the next lowest powers of $\alpha$, one obtains a partial differential equation for $\omega$ from which a correction to the first-order solution can be estimated. Since this correction depends on the first-order solution, it will be discussed in the following section.

### 2.2. *Method of solution when* $\beta = \beta(z)$

In the type of problem that is being considered in this paper, the temperature profile at any instant will be of the general shape shown in figure 1, a practically linear function of depth throughout most of the fluid layer (the gradient may be either positive or negative) with a rapid variation in temperature near the surface. In order to simplify the problem still further, the actual temperature profile will be approximated by two linear segments as shown in figure 2. The subscript $a$ will denote quantities for $z > -\epsilon$, and the subscript $b$ will denote quantities for $z < -\epsilon$. The point of intersection of the two temperature gradients at $\epsilon$ is determined by requiring that the area formed by both the exact and approximate temperature profiles and a vertical line through $T_0$, say, be the same. This ensures that the integrated buoyancy force will be the same in both cases.

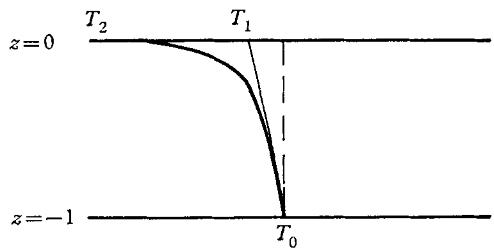


FIGURE 1. General shape of the temperature profile at any time $t$. $T_0$ is the temperature of the lower surface, considered to be a constant. $T_2$ is the time-varying temperature of the upper surface. $T_1$ is the surface temperature as calculated by extrapolation from the temperature and temperature gradient at $z = -1$.

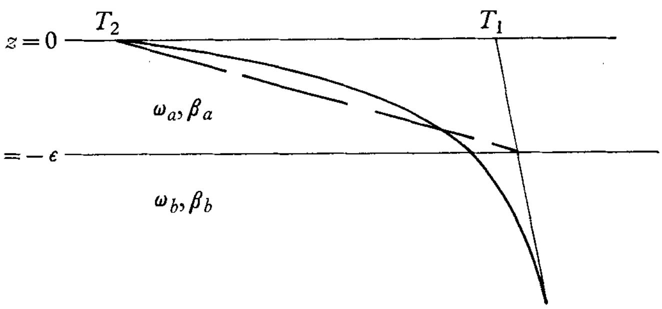


FIGURE 2. Approximate temperature profile consisting of two linear segments. The solid line is the actual temperature profile and the dashed lines are the approximate temperature profiles.

In the regions $z > -\epsilon$ and $z < -\epsilon$, the equations for $\omega$ are now

$$(\partial^2/\partial z^2 - a^2)(\partial^2/\partial z^2 - a^2 - \sigma)(\partial^2/\partial z^2 - a^2 - P\sigma)\,\omega_a = -\beta_a \omega_a, \tag{13}$$

$$(\partial^2/\partial z^2 - a^2)(\partial^2/\partial z^2 - a^2 - \sigma)(\partial^2/\partial z^2 - a^2 - P\sigma)\,\omega_b = -\beta_b \omega_b, \tag{14}$$

where $\beta_a$ and $\beta_b$ are now constants. The solutions to these equations can then be written as

$$\omega_a = \sum_{n=1}^{3} A_n e^{\gamma_n z} + \sum_{n=1}^{3} B_n e^{-\gamma_n z}, \tag{15}$$

$$\omega_b = \sum_{n=1}^{3} C_n e^{\lambda_n z} + \sum_{n=1}^{3} D_n e^{-\lambda_n z}, \tag{16}$$

where the $\gamma_n$'s and $\lambda_n$'s are roots of the equations

$$(\gamma^2 - a^2)(\gamma^2 - a^2 - \sigma)(\gamma^2 - a^2 - P\sigma) = -\beta_a, \tag{17}$$

$$(\lambda^2 - a^2)(\lambda^2 - a^2 - \sigma)(\lambda^2 - a^2 - P\sigma) = -\beta_b. \tag{18}$$

The constants $A_n$, $B_n$, $C_n$, and $D_n$ are determined from the boundary conditions.

The boundary conditions at $z = 0$ and $z = -1$ of course depend on the particular problem, for example whether the fluid is bounded by a rigid or free surface, the manner of heating of the fluid, etc. In order to separate the effects of time-dependent heating from other effects, the simplest boundary conditions will be chosen. These are

$$z = 0, -1: \begin{cases} \omega = 0, & (19) \\ \partial^2\omega/\partial z^2 = 0, & (20) \\ \partial^4\omega/\partial z^4 = 0. & (21) \end{cases}$$

These conditions correspond to the statements that at the boundaries the vertical velocity is zero, no viscous shear stresses are present, and the perturbation temperature is zero, i.e. the temperatures of the surfaces are prescribed. Although these boundary conditions are not physically realistic, they do enable the present analysis to be compared with limiting solutions of other investigations, for which other boundary conditions have also not been investigated. The effect of various boundary conditions has been discussed by Chandrasekhar (1961) and Sparrow, Goldstein & Jonsson (1964).

The boundary conditions at the interface $z = -\epsilon$ are simply that the velocities $u$, $v$, and $w$ are continuous, the viscous shear stresses are continuous, and the perturbation temperature and temperature gradient are continuous. These conditions lead to the requirement that $\omega$ and its first five derivatives with respect to $z$ be continuous at $z = -\epsilon$.

The boundary conditions at $z = 0, -1$ allow $\omega_a$ and $\omega_b$ to be written as

$$\omega_a = \sum_{n=1}^{3} A'_n \sinh \gamma_n z, \tag{22}$$

$$\omega_b = \sum_{n=1}^{3} C'_n \sinh \lambda_n (z+1). \tag{23}$$

The conditions at $z = -\epsilon$ require that

$$\omega_a - \omega_b = 0 \tag{24}$$

and
$$\partial^n\omega_a/\partial z^n - \partial^n\omega_b/\partial z^n = 0 \quad (n = 1, \ldots, 5). \tag{25}$$

In general these conditions cannot be satisfied by solutions to the above equations for arbitrary $\sigma$. The requirement that the equations allow non-trivial solutions satisfying the boundary conditions leads to an eigenvalue problem for $\sigma$.

The condition that (24) and (25) have a non-vanishing solution is that the six-by-six determinant

$$|A_{ij}| = 0, \tag{26}$$

where
$$\begin{aligned} A_{ij} &= \left.\begin{aligned} &\gamma_j^{i-1} \sinh(-\gamma_j \epsilon) \quad \text{if} \quad i \text{ is odd,} \\ &= \gamma_j^{i-1} \cosh(-\gamma_j \epsilon) \quad \text{if} \quad i \text{ is even,} \end{aligned}\right\} \quad (j = 1, 2, 3) \\ &= \left.\begin{aligned} &-\lambda_{j-3}^{i-1} \sinh\{\lambda_{j-3}(1-\epsilon)\} \quad \text{if} \quad i \text{ is odd,} \\ &= -\lambda_{j-3}^{i-1} \cosh(\lambda_{j-3}(1-\epsilon)\} \quad \text{if} \quad i \text{ is even} \end{aligned}\right\} \quad (j = 4, 5, 6). \end{aligned}$$

The method of finding the eigenvalue $\sigma$ when $\beta_a$, $\beta_b$, $a$, and $\epsilon$ are predetermined values is then the following. For the given values of $\beta_a$, $\beta_b$, $a$, and $\epsilon$, the values of the $\gamma_n$'s and $\lambda_n$'s are found from (17) and (18). The parameter $\sigma$ is then varied until, when substituted into (26) with the appropriate $\gamma_n$'s and $\lambda_n$'s, the value of the determinant is approximately equal to zero. All modes for a particular wave-number can be found in this manner. However, since the growth rate is largest for the lowest mode, only the lowest mode will be considered. Once the eigenvalue $\sigma$ is determined, the $A''_n$'s and $C''_n$'s (one of these constants, $A'_1$ say, is arbitrary) can be determined from five of the six relations, (24) and (25).

If we let

$$F(z,t) = f(z,t)\, e^{\alpha\phi(t)} \tag{2.27}$$

the perturbation temperature $f$ can be determined from the equation

$$(\partial^2/\partial z^2 - a^2)\,(\partial^2/\partial z^2 - a^2 - \sigma)\,\omega = f,$$

which can be obtained from (6).

## 3. Results and discussion

As indicated in § 2.1, a complete solution to the problem when $\beta = \beta(z,t)$ requires that a series of problems be solved in which $\beta$ is a function only of depth and not of time. A general solution for arbitrary $\beta(z,t)$ would involve a prohibitive amount of labour. The computations that were made were chosen to illustrate the method of solution and the essential features of the problem.

The growth rate $\sigma$ of disturbances of various wave-numbers $a$ as a function of the Rayleigh number $R$ when the temperature gradient is uniform throughout and independent of time is shown in figures 3 and 4. These calculations were made for the following reasons.

(1) To check the present calculations against previous calculations. The most extensive calculations made previously are those of Morton which are for $P = 1{\cdot}0$ and for small Rayleigh number $R$, where $R = g\alpha d^3(T_2 - T_0)/\kappa\nu$. They agree with the present calculations.

(2) To show the general effect of Prandtl number. For gases, the Prandtl number is approximately 1 and for water at room temperature, the Prandtl number is approximately 7. The calculations were made for these two cases.

(3) To extend previous calculations to high Rayleigh number. For the conditions of the previously mentioned experiment ($T_2 - T_0 = 0{\cdot}5°$ C, $d = 10$ cm), the Rayleigh number is $O(10^8)$ or approximately $10^5 R_c$, where $R_c = 657{\cdot}5$ is the critical Rayleigh number when convective instability begins in the case when the temperature gradient is uniform. Since the method of finding the eigenvalue $\sigma$ is a trial-and-error procedure, these computations are also useful to indicate a reasonable trial solution when $\beta$ is not a linear function of depth.

The general effect of Prandtl number can be seen from figure 3 where the growth rates for both $P = 1{\cdot}0$ and $7{\cdot}0$ have been plotted. The effect of large Prandtl number is to reduce the growth rate significantly (for positive growth rates), although the general variation of $\sigma$ with $a$ is similar. The critical wave-number for marginal stability is not affected by Prandtl number.

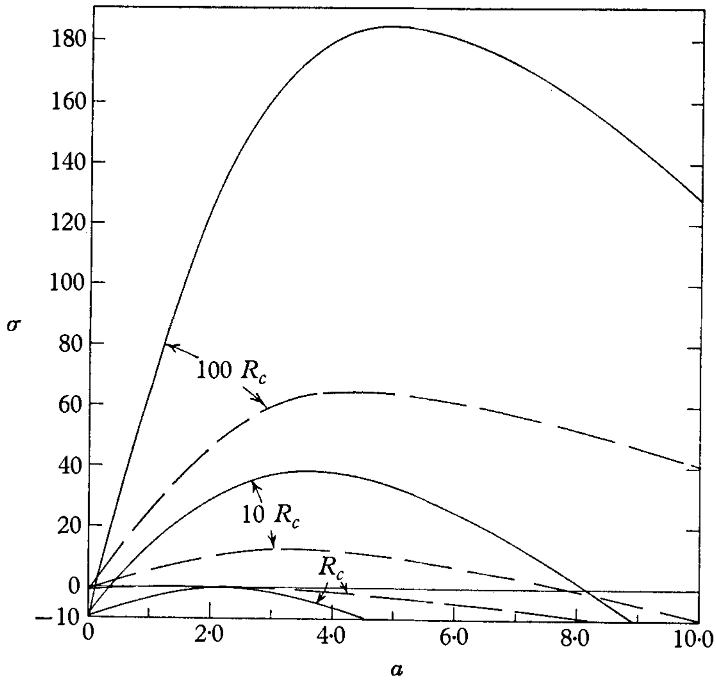


FIGURE 3. The growth rate $\sigma$ of disturbances of wave-number $a$ when the temperature is constant for Rayleigh numbers $R = R_c$, $10R_c$, $100R_c$. The growth rate for $P = 1{\cdot}0$ is given by the solid lines, and for $P = 7{\cdot}0$ by the dashed lines.

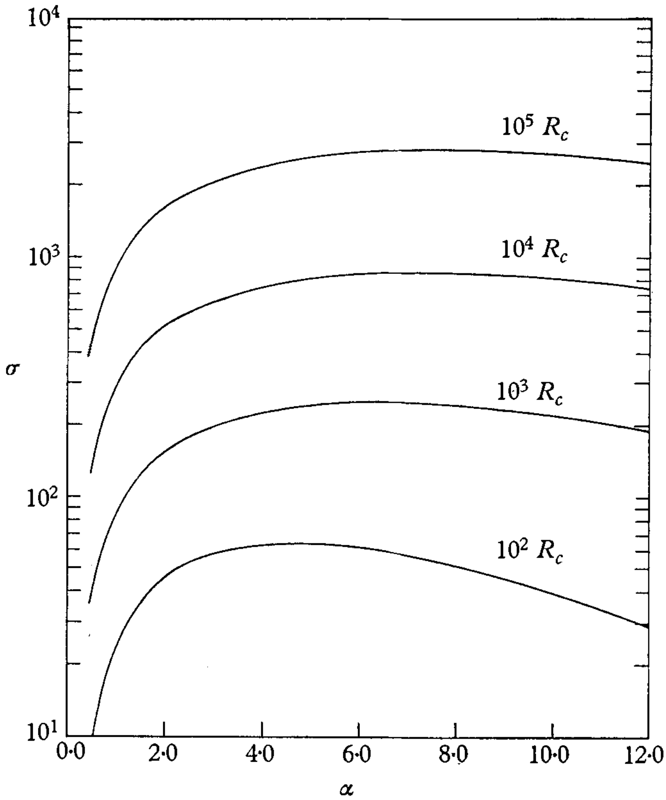


FIGURE 4. The growth rate $\sigma$ of disturbances of wave-number $a$ when the temperature gradient is a constant independent of depth and time for high Rayleigh numbers, $R = 10^2R_c$, $10^3R_c$, $10^4R_c$, $10^5R_c$, and for $P = 7{\cdot}0$.

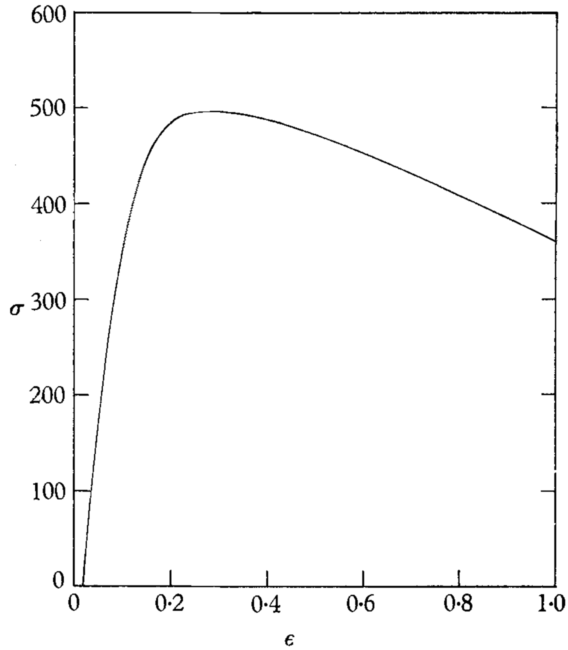


FIGURE 5. The growth rate of disturbances for the wave-number $a = 8{\cdot}0$ as a function of $\epsilon$, the thickness of the layer in which the temperature is changing rapidly. $R_a = 2\times 10^3 R_c$, $R_b = 0{\cdot}1R_c$.

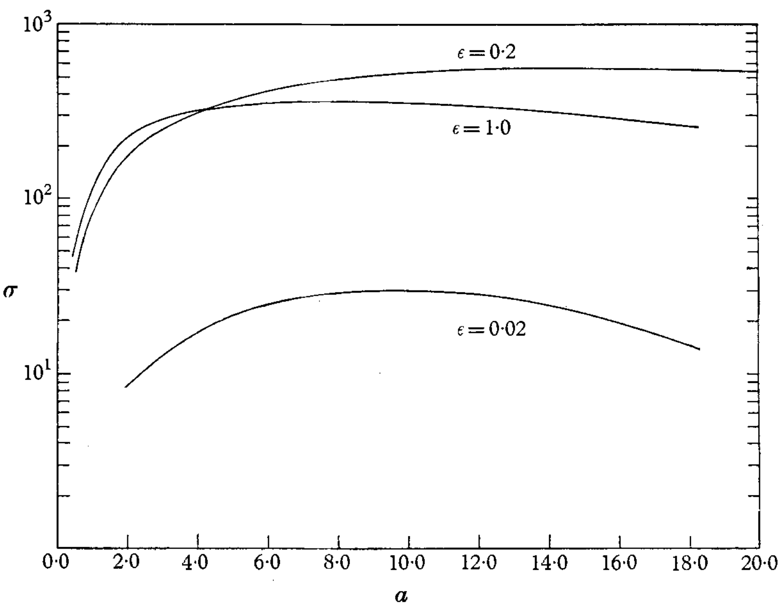


FIGURE 6. The growth rate of disturbances when the temperature gradient consists of two linear segments. $R_a = 2\times 10^3 R_c$, $R_b = 0{\cdot}1R_c$; $\epsilon = 0{\cdot}02,\ 0{\cdot}20,\ 1{\cdot}0$.

The growth rates for large Rayleigh number are shown in figure 4 for $P = 7{\cdot}0$. It can be seen that increasing $R$ increases the wave-number at which the maximum growth rate occurs.

Figures 5 and 6 attempt to show the effect of a non-linear temperature gradient (independent of time) on the growth of disturbances. In all cases, the Prandtl

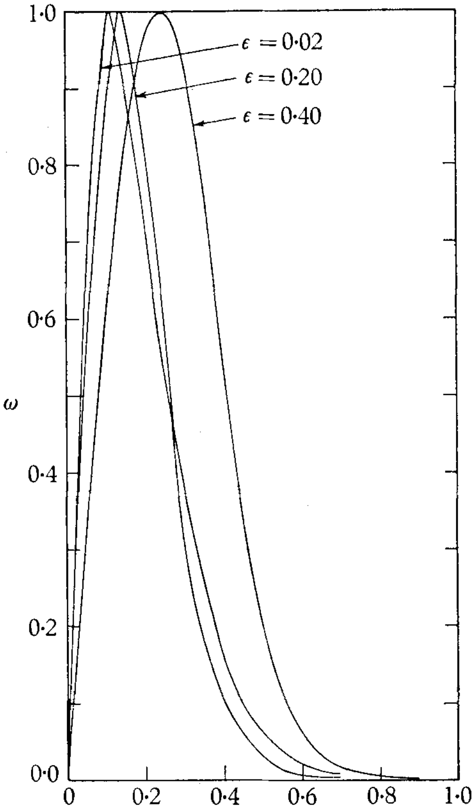


FIGURE 7

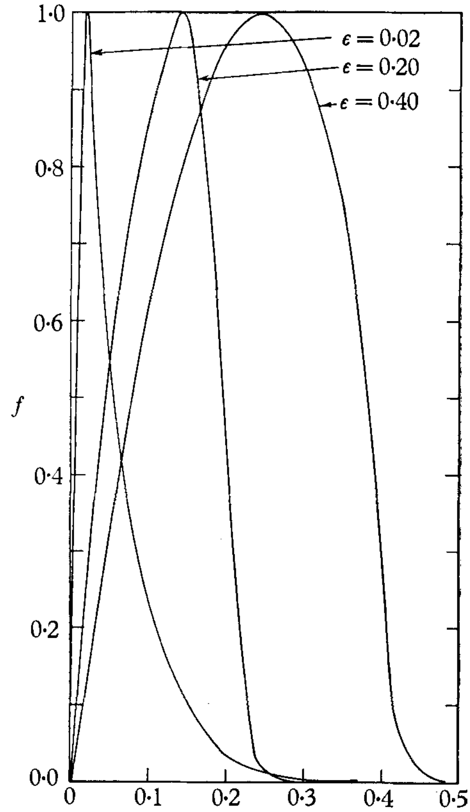


FIGURE 8

FIGURE 7. The vertical velocity $\omega$ as a function of depth $z$ for a non-linear temperature gradient and for the wave-number $a = 12{\cdot}0$. $\epsilon = 0{\cdot}02$, $0{\cdot}20$, $0{\cdot}40$.

FIGURE 8. The perturbation temperature $f$ as a function of depth $z$ for a non-linear temperature gradient and for the wave-number $a = 12{\cdot}0$. $\epsilon = 0{\cdot}02$, $0{\cdot}20$, $0{\cdot}40$.

number is taken to be $7{\cdot}0$ and the overall Rayleigh number is $R_0 = 2 \times 10^3 R_c$. The temperature profile is assumed to consist of two linear segments. A Rayleigh number for each segment can be calculated on the basis of the overall depth $d$. $R_b$ was maintained at $0{\cdot}1R_c$, and $R_a$ was varied so as to keep $R_0$ constant but change the effective depth $\epsilon$ of the upper layer. $R_0 =$ const. implies a constant temperature difference between the two surfaces.

Figure 5 shows the variation of $\sigma$ as a function of $\epsilon$ for a particular wavelength, $a = 8{\cdot}0$. It is to be noted that the maximum growth rate occurs for $\epsilon$ approximately equal to $0{\cdot}25$ and is significantly greater than when the temperature gradient is uniform ($\epsilon = 1{\cdot}0$). This behaviour is typical of all wavelengths and shows that the growth of disturbances when $\beta \neq$ const. may be significantly different from when $\beta =$ const. As the effective depth decreases, the growth rate also decreases and becomes zero at approximately $\epsilon = 0{\cdot}016$.

Figure 6 shows the variation of $\sigma$ as a function of $a$ for three cases, $\epsilon = 0{\cdot}02$, $\epsilon = 0{\cdot}20$, and $\epsilon = 1{\cdot}0$. For $\epsilon = 0{\cdot}02$ and $\epsilon = 0{\cdot}20$, the wave-number for maximum growth rate is significantly larger than when $\epsilon = 1{\cdot}0$.

Figures 7 and 8 show the variation of $\omega$ and $f$ with depth for the conditions $a = 12{\cdot}0$ and $\epsilon = 0{\cdot}40$, $0{\cdot}20$ and $0{\cdot}02$. Of course, when $\epsilon = 1{\cdot}0$, both $\omega$ and $f$ have a simple sine dependence on $z$. Decreasing $\epsilon$ decreases the depth for which either $\omega$ or $f$ is maximum. It can be seen that $\omega$ and $f$ are maximum at different depths for the same $\epsilon$, i.e. the velocity can penetrate into the lower stable layer more easily than the temperature. By tracing the path of illuminated particles the penetration of the velocity could be observed experimentally. The temperature gradient (or density gradient) could be measured by means of a Schlieren system.

The behaviour of the solution as implied by figures 5–8 can be interpreted in more physical terms. When the unperturbed temperature gradient is large in a narrow layer, i.e. when $\epsilon$ is small, it is to be expected that the boundary conditions at $z = -1$ should not affect the growth of disturbances. What should be important are the local temperature gradients and conditions. From figure 5 it can be seen that $\sigma = 0$ for $\epsilon = 0{\cdot}016$. This corresponds to a Rayleigh number of approximately 300 when calculated on the basis of the depth of this narrow layer. The critical Rayleigh number of this layer when it is specified that the vertical velocity is zero at the interface is $657{\cdot}5$. Removing this restriction on the velocity increases the freedom of motion of the fluid and decreases its stability. As $\epsilon$ increases, the Rayleigh number increases, since $R \sim \epsilon^3$, decreasing the stability even though the temperature gradient decreases. Eventually however the boundary conditions at $z = -1$ become important and increase the stability of the fluid.

From the previous calculations one can find $\omega(z, t)$ and $\sigma(t)$ for any constant of time $t$. The solution for $w$ can then be found from (12) and (10) once $\epsilon(t)$ has been determined for each particular problem considered. Note that to the first approximation, the variation of $w$ with depth is the same as the case when $\beta$ is independent of time, although the magnitude of $w$ is different, being given by the factor $\epsilon^{\alpha\phi(t)}$.

The next higher approximation gives a correction term proportional to

$$[\sigma-(P+1)(\gamma^2-a^2)/2P]^{-\frac{1}{2}}.$$

For the fastest growing disturbances, in which we are mainly interested, this correction is approximately $\sigma^{-\frac{1}{2}}$. This correction is similar to the one found for ordinary differential equations (Jeffreys 1962). Of course this correction is large when $\sigma$ is small, i.e. when the fluid is near marginal stability and disturbances are growing slowly. However this only occurs for small time when the unperturbed temperature has penetrated only into a very thin layer. If disturbances occur for larger time, when the fluid is highly unstable, the present theory should give an adequate description of the growth.

The calculations have been made for the case when the fluid in the lower layer is moderately stable, $R_b = 0{\cdot}1R_c$. Differences can be expected when the lower layer is marginally stable, $R_b \simeq R_c$, or when the lower layer is very stable, $R_b \ll R_c$. In the first case the fluid from the upper layer would be expected to penetrate more deeply into the lower layer while the reverse would be true in the second case.

Implicitly it has been assumed throughout that $\sigma$ is real. This has been proved by Morton for the case of a temperature gradient that is negative throughout the fluid. However, $\sigma$ may be complex when the temperature gradient is positive at any point. In this case the eigenvalue problem becomes much more difficult but is solvable in the same manner as that presented above.

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